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A NONLINEAR CONSERVATION LAW WITH MEMORY

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ABSTRACT

In this paper we study a history-boundary value problem for a nonlinear conservation with fading memory in one space dimension. The motivation for studying this problem is an earlier work by C. M. Dafermos and the author concerning the motion of a nonlinear, one-dimensional viscoelastic body. Using a variant of an energy method applied to the viscoelastic problem it is shown that under physically reasonable assumptions the nonlinear conservation law has a unique, classical solution (global in time), provided the data are sufficiently smooth and "small" in a suitable norm; moreover, the solution and its first order derivatives decay to zero as $t \to \infty$. The proof illustrates the versatility of the energy method combined with frequency domain techniques for Volterra operators.

A preliminary analysis based on current work of R. Malek-Madani and the author is presented concerning the development of singularities in smooth solutions of the conservation law (in finite time) for sufficiently "large" smooth data; under special assumptions it is shown that such singularities necessarily develop. The hope is to apply such a procedure to the viscoelastic problem.

AMS (MOS) Subject Classifications: 35L65, 35L67, 45G10, 45K05, 45D05, 45M10, 45M99, 47H10, 47H15, 47H17, 73F15, 73H10, 76N15

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SIGNIFICANCE AND EXPLANATION

Problems arising in continuum mechanics can often be modeled by quasilinear hyperbolic systems in which the characteristic speeds are not necessarily constant. Such systems have the property that waves may be amplified and solutions that were initially smooth may develop discontinuities ("shocks") in finite time. Of particular interest are situations in which the destabilizing mechanism arising from nonlinear effects can coexist and compete with dissipative effects.

In certain cases dissipation is so powerful that waves cannot break and solutions remain smooth for all time. A more interesting situation arises when the amplification and decay mechanisms are nearly balanced so that the outcome of their confrontation cannot be predicted at the outset. A dimensional analysis indicates that the breaking of waves develops on a time scale inversely proportional to wave amplitude, whereas dissipation proceeds at a rate roughly independent of amplitude. Therefore, when the initial data are small it might be expected that the dissipation effects would prevail and waves would not break. Results of this type have been obtained by Nishida [17] for a model problem concerning a quasilinear second order wave equation in one space dimension. Nishida's analysis uses the method of Riemann invariants and is therefore restricted to one space dimension. Using energy methods, Matsumura [16] has studied the case of more than one space dimension, and was able to prove the existence of smooth solutions for all time for quasilinear hyperbolic systems with frictional damping. The necessity of the presence of some form of frictional damping to avoid the formation of "shocks" in finite time follows from a fundamental result of Lax [10]. An assumption of this theory is that the constitutive relation characterizing the nonlinear partial differential operator is convex, but in the case of non-convex functions, as may arise in problems in nonlinear elasticity, similar 'blow-up' results have been obtained by MacCamy and Mizel [12]; general results for nonlinear hyperbolic equations have been obtained by John [8] and Kleinerman and Majda [9].

A different and subtler dissipative mechanism is induced by memory effects of elastico-viscous materials. Dafermos and Nohel [5] have recently developed and analyzed a one-dimensional nonlinear model for the homogeneous extension of an elastico-viscous rod whose ends are free of traction. This equation, which is a quasilinear hyperbolic integrodifferential equation of Volterra type, is based on the assumption that the stress is a particular nonlinear functional of the strain involving two assigned constitutive relations, and an assigned stress-strain relaxation function. Shock solutions are known to occur for this model in the special case in which the stress-strain relaxation function is identically constant. But, by combining energy methods with frequency-domain techniques for nonlinear Volterra equations, it has been shown in [5] that the nonlinear integro-differential equation which describes the motion has a unique, smooth solution for all time <u>provided</u> the given data (history of motion and external body force) are sufficiently smooth and "small." The decay properties of the solution and of its derivatives have also been analyzed. The hypotheses for this theory include all currently used constitutive and stress-strain relaxation functions. This work generalizes studies by MacCamy [11] who used the method of Riemann invariants, and by Dafermos and Nohel [4] and Staffans [20] who used the energy method; in these the stress-strain functional involved a single assigned constitutive function. The energy method developed for these problems was used recently by Slemrod [18] to study the evolution of smooth solutions of a mathematical model for thermoelasticity where the dissipation arose through the influence of thermal damping.

A natural and difficult question for the viscoelastic problems is: do shock solutions develop (i.e. do the waves break in finite time) as the smooth data become sufficiently large? This question was answered in the affirmative by Slemrod [19]

for the aforementioned thermoelastic problem. For the more subtle viscoelastic model the problem is far from solved in its full generality.

The purpose of this paper is to study a simpler model problem consisting of a nonlinear conservation law with memory ((1.1) below), together with a prescribed history function for $t < 0$, and a prescribed boundary condition. This problem, which is of independent interest, is of first order (while the viscoelastic problem is of second order); however it displays the crucial features of the more complex problem. Namely, it is quasilinear hyperbolic and in the absence of the memory term the model problem reduces to Burger's equation which motivated the above mentioned theory of Lax on the formation of shocks in finite time.

In this paper we modify the energy method developed by Dafermos and the author [5], and obtain the global existence of smooth solution for sufficiently smooth and "small" data of the model conservation law with memory.

We then formulate an approach to studying the formation of singularities (shocks) in smooth solutions in finite time; an advantage of this approach is that whenever it is applicable, physically meaningful entropy conditions are satisfied by the shock solutions. We present one such result covering a wide class of problems. The ultimate objective is to extend these methods to systems of conservation laws with memory, and eventually to the viscoelastic problem.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NONLINEAR CONSERVATION LAW WITH MEMORY

J. A. Nohel

1. Introduction. In this paper we study the model nonlinear Volterra functional differential equation (with infinite memory)

$$(1.1)\qquad u_t + \varphi(u)_x + \int_{-\infty}^{t} a'(t-\tau)\psi(u(\tau,x))_x d\tau = f(t,x) \qquad (-\infty < t < \infty,\ 0 \le x \le 1)\ ,$$

where $\varphi,\psi : R \to R$ are given smooth constitutive functions, $a : [0,\infty) \to R$ is a given memory kernel, and $f : R \times [0,1] \to R$ is a given function representing an external force; subscripts denote partial derivatives and $' = d/dt$. The motivation and the assumptions under which (1.1) is studied are provided by the more complex physical problem of the extension of a finite, homogeneous, elastoviscous body moving under the action an assigned body force. The viscoelastic problem, formulated in Section 2, was recently studied by Dafermos and Nohel [5]; references to earlier literature are given in Section 2.

The model problem (1.1), which is of independent interest, is simpler in that it is of first order, while the equation of motion (2.8) below, is of second order; otherwise (1.1) incorporates the interesting features of (2.8). The most important of these is the following. If $\psi \equiv 0$, $f \equiv 0$, (1.1) reduces to Burger's equation (conservation law of gas dynamics):

$$(1.2)\qquad u_t + \varphi(u)_x = 0\ ;$$

note that (1.2) is quasilinear and hyperbolic. It is classical, see Lax [10], that the Cauchy problem consisting of Burger's equation and the initial condition $u(0,x) = u_0(x)$, $x \in R$, does not, in general, possess a classical smooth solution, no matter how smooth (and "small") one takes the initial datum u_0; if φ is convex Lax [10] has shown that the solution develops a singularity (shock) in finite time due to the crossing of characteristics. More precisely, if $u_0'(x) > 0$, u_0 smooth, and φ is convex, (1.2) has a

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smooth solution for all $t > 0$, while if $u_0'(x_0) < 0$ for some x_0, the characteristics of (1.2) will cross and shocks will develop in finite time; a similar result also holds for systems of conservation laws in one space dimension. If φ is not convex, e.g. in nonlinear elasticity, similar results have been established by MacCamy and Mizel [12]; for recent closely related literature see also general results by John [8], Kleinerman and Majda [9], Malek-Madani [13].

The purpose of this paper is: First, in Section 2 we formulate the problem of motion of a nonlinear viscoelastic body as analysed recently by Dafermos and Nohel [5] using a combination of energy methods and properties of strongly positive kernels. Second, under assumptions motivated by the viscoelastic problem, we show in Sections 3 and 4 that equation (1.1), together with an assigned periodic boundary condition and an assigned smooth history

$$u(t,x) = v(t,x), \qquad -\infty < t \leq 0, \qquad 0 \leq x \leq 1 ,$$

which satisfies (1.1) for $t \leq 0$, possesses a unique classical solution for all $t > 0$, provided the history function v and the forcing term f are sufficiently smooth and "small" in suitable norms. This result, in which the same strategy as in [5] is used, exhibits the dissipative character of the integral term in (1.1); its proof serves to illustrate a general energy technique for hyperbolic, nonlinear Volterra problems. Finally, in Section 5 we formulate the problem of development of singularities in finite time and we present a recent result, analogous to the result for Burger's equation, in an important special case, [15]; the proof will appear elsewhere. The fact that the question of development of singularities for the physically interesting viscoelastic problem remains open, provides the principal motivation for studying the simpler model problem (1.1). It should be noted that the problem discussed in Section 5 is different from the study of weak solutions for the Riemann problem for (1.1) (Greenberg, Ling Hsiao, and MacCamy [6], Dafermos and Ling Hsiao [3]).

2. A Hyperbolic, Nonlinear Volterra Equation in Viscoelasticity. Problems arising in continuum mechanics can often be modeled by quasilinear hyperbolic systems in which the characteristic speeds are not necessarily constant. Such systems have the property that waves may be amplified and solutions that were initially smooth may develop discontinuities ("shocks") in finite time. Of particular interest are situations in which the destabilizing mechanism arising from nonlinear effects can coexist and compete with dissipative effects.

In certain cases dissipation is so powerful that waves cannot break and solutions remain smooth for all time. A more interesting situation arises when the amplification and decay mechanisms are nearly balanced so that the outcome of their confrontation cannot be predicted at the outset. A dimensional analysis indicates that the breaking of waves develops on a time scale inversely proportional to wave amplitude, whereas dissipation proceeds at a rate roughly independent of amplitude. Therefore, when the initial data are small it might be expected that the dissipation effects would prevail and waves would not break. Results of this type have been obtained by Nishida [17] for a model problem concerning a quasilinear second order wave equation in one space dimension. Nishida's analysis uses the method of Riemann invariants and is therefore restricted to one space dimension. Using energy methods, Matsumura [16] has studied the case of more than one space dimension, and was able to prove the existence of smooth solutions for all time for quasilinear hyperbolic systems with frictional damping. Burger's equation (1.2) shows the necessity of the presence of some form of frictional damping to avoid the formation of "shocks". The rather delicate situation of thermal damping in thermoelasticity is discussed by Slemrod [18].

A different and subtler dissipative mechanism is induced by memory effects of elastico-viscous materials. Dafermos and Nohel [5] have recently developed and analyzed a one-dimensional nonlinear model for the homogeneous extension of an elastico-viscous rod whose ends are free of traction. Their simple, one-dimensional, model corresponds to the following constitutive relation, suggested by the theory developed by Coleman and Gurtin [1],

$$\sigma(t,x) = \varphi(e(t,x)) + \int_{-\infty}^{t} a'(t-\tau)\psi(e(\tau,x))d\tau \ , \tag{2.1}$$

where σ is the stress, e the strain, a the relaxation function with $' = d/dt$, and φ, ψ assigned constitutive functions. The relaxation function is normalized so that $a(\infty) = 0$. When the reference configuration is a natural state, $\varphi(0) = \psi(0) = 0$. Experience indicates that $\varphi(e)$, $\psi(e)$, as well as the equilibrium stress

$$\chi(e) \stackrel{\text{def}}{=} \varphi(e) - a(0)\psi(e) \tag{2.2}$$

are increasing functions of e, at least near equilibrium ($|e|$ small). Moreover, the effect of viscosity is dissipative. To express mathematically the above physical requirements, we impose upon $a(t)$, $\varphi(e)$, $\psi(e)$ and $\chi(e)$ the following assumptions:

$$a(t) \in W^{2,1}(0,\infty),\ a(t) \text{ is strongly positive definite on } [0,\infty) \ ; \tag{2.3}$$

$$\varphi(e) \in C^3(-\infty,\infty), \quad \varphi(0) = 0, \quad \varphi'(0) > 0 \ ; \tag{2.4}$$

$$\psi(e) \in C^2(-\infty,\infty), \quad \psi(0) = 0, \quad \psi'(0) > 0 \ ; \tag{2.5}$$

$$\chi'(0) = \varphi'(0) - a(0)\psi'(0) > 0 \ . \tag{2.6}$$

Assumption (2.3), which requires that $a(t) - \alpha \exp(-t)$ be a positive definite kernel on $[0,\infty)$ for some $\alpha > 0$, expresses the dissipative character of viscosity. Smooth, integrable, nonincreasing, convex relaxation functions, e.g.,

$$a(t) = \sum_{k=1}^{K} \nu_k \exp(-\mu_k t), \quad \nu_k > 0, \quad \mu_k > 0 \ , \tag{2.7}$$

which are commonly employed in the applications of the theory of viscoelasticity, satisfy (2.3).

We now consider a homogeneous, one-dimensional body (string or bar) with reference configuration $[0,1]$ of density $\rho = 1$ (for simplicity) and constitutive relation (2.1) which is moving under the action of an assigned body force $g(t,x)$, $-\infty < t < \infty$, $0 \le x \le 1$, with the aids of the rod free of traction. We let $u(t,x)$ denote the displacement of particle x at time t in which case the strain is $e(t,x) = u_x(t,x)$. Thus the equation of motion $\rho u_{tt} = \sigma_x + \rho g$ here takes the form of the nonlinear (hyperbolic) Volterra

functional differential equation

$$(2.8) \qquad u_{tt} = \varphi(u_x)_x + \int_{-\infty}^{t} a'(t-\tau)\psi(u_x)_x d\tau + g, \quad -\infty < t < \infty, \; 0 \le x \le 1 .$$

The physical problem of the motion of a viscoelastic body suggests that the history of the motion of the body up to time $t = 0$ is assumed known, i.e.,

$$(2.9) \qquad u(t,x) = v(t,x), \qquad -\infty < t \le 0, \qquad 0 \le x \le 1 ,$$

where $v(t,x)$ is a given sufficiently smooth function which satisfies equation (2.8) for $t \le 0$, together with appropriate boundary conditions. In order to show that the motion of the viscoelastic bar remains smooth for all $t > 0$, the mathematical task is to determine a smooth extension $u(t,x)$ of $v(t,x)$ on $(-\infty,\infty) \times [0,1]$ which satisfies (2.8) together with assigned boundary conditions, for $-\infty < t < \infty$.

Dafermos and Nohel [5, Theorem 1.1] establish such a global result for the problem (2.8), (2.9) together with homogeneous Neumann boundary conditions

$$(2.10) \qquad u_x(t,0) = u_x(t,1) = 0, \qquad -\infty < t < \infty ;$$

these are shown to be equivalent to the statement: the boundary of the body is free of traction $(\sigma(t,0) = \sigma(t,1) = 0, -\infty < t < \infty, \sigma$ given by (1.1)). Their global result is valid for sufficiently smooth and "small" external body forces g and history functions v. Other boundary conditions and various generalizations are also considered.

The general strategy used in [5] is as follows. First one establishes the existence of a unique smooth local solution u defined on a maximal interval $(-\infty,T_0) \times [0,1]$, with the property that when $T_0 < \infty$ a certain norm of the solution becomes infinite as $t \to T_0^-$; this is done by a fixed point argument (combined with a standard energy method for linear problems) on a suitably chosen abstract space of functions. Second, energy methods are combined with properties of strongly positive kernels to show that due to the viscous dissipation of the integral term in (2.8), the aforementioned norm remains uniformly bounded on the maximal interval, provided the data g and v are sufficiently smooth and small. By standard theory for nonlinear problems this means that $T_0 = +\infty$ and the smooth solution exists globally in t. This part of the analysis involves obtaining a priori

estimates of certain norms of the derivatives (in one space dimension, up to and including order 3) directly from the equation (2.8). It is here that it becomes convenient to use the equivalent form

$$(2.11) \qquad u_{tt} = \chi(u_x)_x + \int_{-\infty}^{t} a(t - \tau)\psi(u_x)_{\tau x}d\tau + g, \quad -\infty < t < \infty, \quad 0 \leq x \leq 1 ,$$

of equation (2.8) (equation (2.11) is obtained from (2.8) by integrating by parts with respect to τ and by using the definition of the equilibrium stress χ); assumption (2.6) plays a crucial role.

For the special case $\psi(e) \equiv \varphi(e)$ various global existence results for (2.8), were established by MacCamy [11], Dafermos and Nohel [4] and Staffans [20]. The assumption $\psi \equiv \varphi$ allows one to invert the linear Volterra integral operator on the right-hand side of (2.8) and thus express $\varphi(u_x)_x$ in terms of $u_{tt} - g$ through an inverse Volterra integral operator using the resolvent kernel associated with a'. One may then transfer time derivatives from u_{tt} to the resolvent kernel via integration by parts. This procedure, which was also discussed by Dafermos [2] in a recent expository paper in a number of special cases, reveals the instantaneous character of dissipation and, at the same time, renders the memory term linear and milder, thus simplifying the analysis considerably. On the other hand, the above approach is somewhat artificial: By inverting the right-hand side of (2.8), one loses sight of the original equation and of the physical interpretation of the derived a priori estimates. More importantly, the physical appropriateness of the restriction $\psi = \varphi$ is by no means clear.

<u>Remark</u>: The present normalization of the kernel a with $a(\infty) = 0$ is different from that in the existing literature (see [4], [11], [20]). The reader should note a', not a, enters the constitutive relation (2.1) as well as the equation of motion (2.8). In the earlier literature in which only the special case $\psi \equiv \varphi$ was studied, the normalization

$$a(t) = a_\infty + A(t) \qquad 0 \leq t < \infty ,$$

$a(0) = 1$, $a_\infty > 0$, $A \in W^{2,1}(0,\infty)$, A strongly positive was used. The normalization used here is crucial for generating the a priori estimates directly from equation (2.11)

(equivalent to (2.8)). The reader should note that the present normalization and (2.6) imply that $0 < a(0) < 1$, if $\varphi \equiv \psi$.

The question of the development of singularities of solutions of (2.8) in finite time for sufficiently "large", smooth data (which have been observed for viscoelastic bodies [1]) is under active study. Some partial results with $\psi \equiv \varphi$ in (2.8) have been obtained by Hattori [7]; for a viscoelastic fluid (see Slemrod [19]). However, in the general case of (2.8) the problem is far from settled. For this reason we believe that the approach in Section 5 for the considerably simpler problem (1.1) is particularly useful and suggestive.

3. <u>A Conservation Law with Fading Memory</u>. We study the model nonlinear, history-boundary value problem

$$(3.1)\qquad u_t + \varphi(u)_x + \int_{-\infty}^{t} a'(t-\tau)\psi(u(\tau,x))_x d\tau = f(t,x) \qquad (-\infty < t < \infty,\ 0 \le x \le 1)\ ,$$

subject to the periodic boundary condition

$$(3.2)\qquad u(t,0) = u(t,1)\ ;$$

the history of the solution u is assumed to be known up to time $t = 0$, i.e.

$$(3.3)\qquad u(t,x) = v(t,x) \qquad (-\infty < t \le 0,\ 0 \le x \le 1)\ ,$$

where v is a given smooth $(C^1((-\infty,0] \times [0,1]))$ function which satisfies (3.1), (3.2) for $t \le 0$. In (3.1) $\varphi,\psi : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \times [0,1] \to \mathbb{R}$, and $a : [0,\infty) \to \mathbb{R}$ are given functions satisfying assumptions analogous to those for the viscoelastic problem outlined in Section 2. Our task in this section is to determine a smooth extension $u(t,x)$ of $v(t,x)$ on $(-\infty,\infty) \times [0,1]$ which satisfies (3.1) and (3.2), and to study the asymptotic properties of u as $t \to \infty$. In order to do this the history v and the forcing function f will have to be taken sufficiently smooth and "small" in suitable norms.

The requirement that the history function v should satisfy (3.1) and (3.2) for $t < 0$ is motivated by the viscoelastic problem described in Section 2. The history value problem in which the function v is sufficiently smooth for $t \le 0$ (but need not satisfy (3.1)) is also of interest, and can be studied by the same methods (see further remarks below).

The basic assumptions for the global existence theory are as follows. Concerning the constitutive functions φ,ψ:

$$(c)\qquad \varphi,\psi \in C^2(\mathbb{R}),\ \ \varphi(0) = \psi(0) = 0,\ \ \varphi'(0) > 0,\ \ \psi'(0) > 0\ ;$$

concerning the kernel a:

$$(a)\qquad a \in W^{2,1}(0,\infty) \text{ and } a \text{ is strongly positive on } [0,\infty)\ ;$$

concerning the forcing term f:

$$
\text{(f)} \qquad \begin{cases} f, f_t, f_x \in C[(-\infty,\infty); L^2(0,1)] \cap L^2[(-\infty,\infty); L^2(0,1)] \ , \\ f(t,x) = f_1(t,x) + f_2(t,x) \ , \\ f_{1tt}, f_{2tx} \in L^2[(-\infty,\infty); L^2(0,1)] \ , \\ \\ \int_0^1 f(t,x)\,dx = 0 \qquad (-\infty < t < \infty) \ , \end{cases}
$$

and to measure the "size" of the forcing term we define

$$
F = \sup_{(-\infty,\infty)} \int_0^1 \{f^2 + f_t^2 + f_x^2\}(t,x)\,dx + \int_{-\infty}^{\infty} \int_0^1 \{f^2 + f_t^2 + f_x^2 + f_{1tt}^2 + f_{2tx}^2\}\,dx\,dt \ ;
$$

concerning the history v:

$$
\text{(H)} \qquad \begin{cases} v, v_t, v_x, v_{tt}, v_{tx}, v_{xx} \in C[(-\infty,\infty); L^2(0,1)] \cap L^2[(-\infty,\infty); L^2(0,1)], \\ \\ \int_0^1 v(t,x)\,dx = 0 \qquad (-\infty < t \leq 0) \ , \\ \\ \text{and } v \text{ satisfies (3.1), (3.2) for } t \leq 0 \ . \end{cases}
$$

Analogous to the "equilibrium stress" for the viscoelastic body we define the constitutive function $\chi : R \to R$ by

$$
\chi(\cdot) = \varphi(\cdot) - a(0)\psi(\cdot) \ ,
$$

and we assume that

$$
\chi'(0) = \varphi'(0) - a(0)\psi'(0) > 0 \ . \tag{3.4}
$$

The following equation, obtained from (3.1) by carrying out an integration by parts with respect to τ, will play a crucial role in the analysis:

$$
u_t + \chi(u)_x + \int_{-\infty}^{t} a(t-\tau)\psi(u(\tau,x))_{x\tau}\,d\tau = f(t,x) \qquad (-\infty < t < \infty,\ 0 \leq x \leq 1) \ . \tag{3.5}
$$

It is clear that the problem consisting of (3.5), (3.2), (3.3) is equivalent to the original problem (3.1), (3.2), (3.3).

Upon setting

$$(3.6)\qquad h(t,x) = -\int_{-\infty}^{0} a'(t-\tau)\psi(v(\tau,x))_x d\tau + f(t,x) \quad (0 \le t < \infty,\ 0 \le x \le 1)$$

$$(3.7)\qquad u_0(x) = v(0,x) \qquad (0 \le x \le 1)\ ,$$

the history-boundary value problem (3.1)-(3.3) reduces to the intial-boundary value problem

$$(3.8)\qquad \begin{cases} u_t + \varphi(u)_x + \int_0^t a'(t-\tau)\psi(u(\tau,x))_x d\tau = h(t,x) & (0 < t < \infty,\ 0 \le x \le 1) \\ u(t,0) = u(t,1) & (0 \le t < \infty) \\ u(0,x) = u_0(x) & (0 \le x \le 1)\ , \end{cases}$$

where $\int_0^1 h(t,x)dx = 0$ $(0 \le t < \infty)$ [use $v(t,0) = v(t,1)$, and assumptions (f)]. Conversely a solution of (3.8), where $\int_0^1 u_0(x)dx = 0$, can be shown to solve (3.1)-(3.3) by constructing a smooth function $v(t,x)$ on $(-\infty,0] \times [0,1]$, satisfying (3.2), $\int_0^1 v(t,x)dx = 0$ $(-\infty < t \le 0)$, and also requiring $v(0,x) = u_0(x)$, as well as

$$(3.9)\qquad \begin{cases} v_t(0,x) = -\varphi(u_0(x))_x + h(0,x) \overset{\text{def}}{=} u_1(x) \\ v_{tt}(0,x) = -\varphi''(u_0(x))u_0'(x)u_1(x) - \varphi'(u_0(x))u_1'(x) \\ \qquad\qquad - a'(0)\psi(u_0(x))_x + h_t(0,x)\ , \end{cases}$$

and defining

$$(3.10)\qquad f(t,x) = \begin{cases} v_t + \varphi(v)_x + \int_{-\infty}^{t} a'(t-\tau)\psi(v(\tau,x))_x d\tau & (-\infty < t \le 0,\ 0 \le x \le 1) \\ h(t,x) + \int_{-\infty}^{0} a'(t-\tau)\psi(v(\tau,x))_x d\tau & (0 < t < \infty,\ 0 \le x \le 1)\ ; \end{cases}$$

the requirements (3.9) insure that f defined by (3.10) has the necessary smoothness properties across $t = 0$ required in the existence theory.

The main global result is

Theorem 3.1. Let the assumptions (a), (c), and (3.4) be satisfied. There exists a constant $\mu > 0$ such that for every forcing term f satisfying assumptions (f) with

$F \leq \mu^2$, and for any history function v on $(-\infty,0] \times [0,1]$ satisfying assumptions (H), there exists a unique function u on $(-\infty,\infty) \times [0,1]$ with $u, u_t, u_x, u_{tt}, u_{tx}, u_{xx} \in C[(-\infty,\infty); L^2(0,1)] \cap L^2[(-\infty,\infty); L^2(0,1)]$ satisfying (3.1)-(3.3), as well as $\int_0^1 u(t,x)\,dx = 0 \quad (-\infty < t < \infty)$. Moreover,

$$u(t,x),\ u_t(t,x),\ u_x(t,x) \to 0 \quad \text{as} \quad t \to \infty, \tag{3.11}$$

uniformly on $[0,1]$.

The proof of Theorem 3.1 will be given in Section 4. It uses the general strategy developed in [5], although there are technical differences in details. One first establishes the existence of a local solution u on a maximal interval $(-\infty,T_0) \times [0,1]$, with the property that when $T_0 < \infty$ a certain norm of u becomes infinite as $t \to T_0^-$ (see Proposition 4.1 below). One then uses a combination of energy methods and properties of strongly positive kernels to show that the integral in (3.1) exerts a dissipative effect resulting in the aforementioned norm remaining uniformly bounded, independent of T_0, provided the constant μ in Theorem 3.1 is sufficiently small. Thus, in particular $T_0 = +\infty$ and the smooth solution exists globally.

Remark 3.2. It follows by standard regularity techniques that the solution u of (3.1), (3.2), (3.3) is C^1 smooth on $\mathbb{R} \times [0,1]$.

Remark 3.3. A result similar to Theorem 3.1 (established by the same methods) holds for the boundary-initial value problem

$$\begin{cases} u_t + \varphi(u)_x + \int_0^t a'(t-\tau)\psi(u(\tau,x))_x\,d\tau = h(t,x) & (0 < t < \infty,\ 0 \leq x \leq 1)\ , \\ u(t,0) = u(t,1) & (0 \leq t < 1)\ , \\ u(0,x) = u_0(x) & (0 \leq x \leq 1,\ u_0(0) = u_0(1))\ ; \end{cases} \tag{3.12}$$

here φ, ψ, and a satisfy the assumptions of Theorem 3.1, h defined on $[0,\infty) \times [0,1]$ satisfies assumptions (f) modified in the obvious way, and the initial datum $u_0 \in H^2(0,1)$, with $\int_0^1 u_0(x)\,dx = 0$. For the theorem to hold one must require that $\|u_0\|_{H^2}$, and a suitable norm of h are sufficiently small, as can be seen from a detailed examination of the corresponding estimates. (See Remark 4.3 below).

Remark 3.4. The requirement that the history v satisfies (3.1), (3.2) for $t \leq 0$, and the condition: the constant μ (where $F \leq \mu^2$) of Theorem 3.1 is "small", imply that the history function v, as well as the forcing term f in (3.1), are both small in suitable norms.

Remark 3.5. If $\varphi \equiv \psi$ in (3.1), Theorem 3.1 can be applied without any change, provided $0 < a(0) < 1$ (in order that (3.4) is satisfied). However, in this special case the somewhat different energy techniques of Dafermos and Nohel [4], or of Staffans [20], or the method of Riemann invariants of MacCamy [11] can also be used.

Remark 3.6. It will follow from the proof of Theorem 3.1 in Section 4 that because the problem (3.1)-(3.3) is in one space dimension and on a finite space interval it is sufficient to obtain global estimates for the derivatives (in this case estimates of u_{tt}, u_{tx}, u_{xx}, because (3.1) is of first order) in the $L^\infty(L^2(0,1))$ and $L^2(L^2(0,1))$ norms, and then to make use of the Poincaré inequality to estimate lower order terms. However, one cannot apply this method to obtain global estimates for the pure Cauchy problem consisting of the Volterra equation

$$(3.13) \qquad u_t + \varphi(x)_x + \int_0^t a'(t-\tau)\psi(u(\tau,x))_x d\tau = h(t,x) \qquad (0 < t < \infty,\ x \in \mathbb{R}) ,$$

together the initial condition $u(0,x) = u_0(x)$, $x \in \mathbb{R}$, because $x \in \mathbb{R}$ and the Poincaré inequality does not apply. For the local existence result (analogue of Proposition 4.1) one can circumvent this difficulty. The same comments apply to the analogous pure Cauchy problem associated with the second-order nonlinear Volterra equation (2.8). If $\phi \equiv \psi$ in (3.13) or (2.8) the analogous pure Cauchy problems can be treated by either the methods of [4], [11] or of [20], because global estimates of u and of the derivatives of u in appropriate norms are obtained successively from the equations. However, if $\phi \not\equiv \psi$ in (3.13) or the pure Cauchy problem associated with (2.8) these mathematically interesting problems remain to be tackled.

4. Proof of Theorem 3.1; a. Local Theory. This is carried for the boundary-initial value problem (which was shown to be equivalent to the history value problem (3.1)-(3.4) in Section 3 for a suitable choice of the forcing term f - see (3.10)):

$$(4.1)\quad \begin{cases} u_t + \varphi(u)_x + \int_0^t a'(t-\tau)\psi(u(\tau,x))_x d\tau = h(t,x) & (0 < t < \infty,\ 0 \le x \le 1)\ , \\ u(t,0) = u(t,1), \quad 0 \le t < \infty\ , \\ u(0,x) = u_0(x), \quad x \in [0,1]\ . \end{cases}$$

We make the following assumptions:

$$a', a'' \in L^1(0,\infty),\quad \varphi,\psi \in C^2(\mathbb{R}),\quad \varphi(0) = \psi(0) = 0,\quad \psi'(0) > 0\ ,$$

there exists a constant $\kappa > 0$ such that $\varphi'(\xi) \ge \kappa > 0$ $(\xi \in \mathbb{R})$,

$h : [0,\infty) \times [0,1] \to \mathbb{R}$, $h(t,x) = h_1(t,x) + h_2(t,x)$, $h, h_t, h_x \in C([0,\infty);L^2(0,1))$, $\int_0^1 h(t,x)dx = 0$ $(t \in [0,\infty))$, $h_{1tt}, h_{2tx} \in L^2([0,\infty);L^2(0,1))$, and $u_0 \in H^2(0,1)$, $\int_0^1 u_0(x)dx = 0$. The reader will note that the full strength of assumptions (a) and assumption (3.4) are not used in Proposition 4.1 below; on the other hand φ' is now required to be bounded away from zero (compare with assumptions (c)).

Proposition 4.1. Under the above assumptions there exists a $0 < T_0 \le \infty$ and a unique function $u \in C^1([0,T_0) \times [0,1]$ with $u_{tt}, u_{tx}, u_{xx} \in C([0,T];L^2(0,1))$ for every $0 < T < T_0$, such that u satisfies (4.1) on $[0,T_0) \times [0,1]$ and $\int_0^1 u(t,x)dx = 0$. Moreover, if $T_0 < \infty$

$$(4.2)\qquad \sup_{t \to T_0^-} \int_0^1 [u^2(t,x) + u_t^2(t,x) + \cdots + u_{xx}^2(t,x)]dx = +\infty\ .$$

It is clear that with h defined (3.6), u_0 by (3.7) the problem (3.1)-(3.3) satisfies the assumptions of Proposition 4.1.

Remark 4.2. A similar result holds for the pure Cauchy problem associated with (4.1), i.e. no boundary condition is specified and $u(0,x) = u_0(x)$, $u_0 \in H^2(\mathbb{R})$; however, the function space $X(M,T)$ below must be specified differently.

The proof of Proposition 4.1 is very similar to that of Theorem 2.1 in [5], and will only be sketched. Let $M > 0$, $T > 0$ and let $X(M,T)$ denote the set of functions

$w(t,x)$ on $[0,T] \times [0,1]$ with $w, w_t, w_x, w_{tt}, w_{tx}, w_{xx} \in C([0,T;L^2(0,1))$, $w(t,0) = w(t,1)$, $w(0,x) = u_0(x)$, $\int_0^1 w(t,x)dx = 0$, $\int_0^1 w_t(t,x)dx = 0$, $0 \le t \le T$, and

$$\sup_{[0,T]} \int_0^1 [w_{tt}^2(t,x) + w_{tx}^2(t,x) + w_{xx}^2(t,x)]dx \le M^2 . \tag{4.3}$$

It follows from the Poincaré-type inequalities (see application of Lemmas A.1 and A.2 in Appendix) and from (4.3) that if $w \in X(M,T)$ then

$$w^2(t,x) + w_x^2(t,x) + w_t^2(t,x) \le M^2 \quad (0 \le t \le T,\ 0 \le x \le 1) . \tag{4.4}$$

Let $S : X(M,T) \to C^1([0,T] \times [0,1]$ be the mapping which carries $w \in X(M,T)$ into the solution of the linear problem consisting of

$$u_t + \varphi'(w)u_x = -\int_0^t a'(t-\tau)\psi(w(\tau,x))_x d\tau + h(t,x) \quad (0 < t \le T,\ 0 \le x \le 1) , \tag{4.5}$$

and of the boundary and initial conditions in (4.1). It is clear that a fixed point of S is a solution of (4.1) on $[0,T] \times [0,1]$. Also note $\varphi'(w)$ is $W^{1,\infty}$ smooth and $\varphi'(w)_t$ and $\varphi'(w)_x$ are in $L^\infty([0,T];L^2(0,1))$. Moreover, if $g(t,x)$ denotes the right-hand side of (4.5), $g(t,x) = g_1(t,x) + g_2(t,x)$, then g satisfies the same assumptions as h does, and by fairly standard theory for linear problems $u_{tt}, u_{tx}, u_{xx} \in C([0,T];L^2(0,1))$; embedding type arguments then yield that $u \in C^1([0,T] \times [0,1])$. Thus it suffices to show that the map S has a unique fixed point u in $X(M,T)$. Once this has been demonstrated it follows from the assumption $\int_0^1 h(t,x)dx = 0$, $t \in [0,1]$, the equation in (4.1), and from the boundary condition that $\frac{\partial}{\partial t}\int_0^t u(t,x)dx = 0$ for $t \in [0,T]$; since $\int_0^1 u_0(x)dx = 0$, one then also has $\int_0^1 u(t,x)dx = 0$, $t \in [0,T]$.

The remainder of the proof of Proposition 4.1 is completed in the following steps: (i) analogous to the proof of Lemma 2.1 in [5] (here the argument is shorter), use the standard energy method for linear problems to show that when M is sufficiently large and $T > 0$ is sufficiently small, S maps $X(M,T)$ into itself; (ii) define the metric

$$\rho(u,\bar u) = \max_{[0,T]} \{\int_0^1 [(u_t - \bar u_t)^2 + (u_x - \bar u_x)^2](t,x)dx ,$$

where $u, \bar u \in X(M,T)$; by the lower semicontinuity of norms in a Banach space, $X(M,T)$ is complete under ρ; analogous to Lemma 2.2 in [5] show that S is a strict contraction on $X(M,T)$ under the metric ρ; (iii) by Banach's fixed point theorem the map S has a unique fixed point $u \in C^1([0,T] \times [0,1])$ for M sufficiently large and T sufficiently small, which solves (4.1); the existence of the maximal interval of validity $[0,T_0) \times [0,1]$ of the solution u satisfying (4.2) is established in a standard manner (see [5]). This completes the sketch of the proof of Proposition 4.1.

b. Global Theory. By the constitutive assumptions (c) and (3.4) there exist $\delta > 0$, $\kappa > 0$ such that

(4.6) $$\varphi'(\xi) \geq \kappa,\ \psi'(\xi) \geq \kappa,\ \chi'(\xi) \geq \kappa \qquad (|\xi| \leq \delta) .$$

If necessary modify φ outside $[-\delta,\delta]$ such that $\varphi \in C^2(\mathbb{R})$ and $\varphi'(\xi) \geq \kappa$ $(\xi \in \mathbb{R})$. To prove the existence of the global solution u on $(-\infty,\infty) \times [0,1]$ of the history value problem (3.1)-(3.3) asserted in Theorem 3.1, let u be the unique solution on the maximal interval $(-\infty,T_0) \times [0,1]$ guaranteed by Proposition 4.1, and assume that $0 < T_0 < \infty$.

For $0 < T < T_0$ let

(4.7) $$U(T) = \sup_{(-\infty,T]} \int_0^1 [u^2(t,x) + u_t^2(t,x) + u_x^2(t,x) + \cdots + u_{xx}^2(t,x)]dx$$
$$+ \int_{-\infty}^T \int_0^1 [u^2 + u_t^2 + u_x^2 + \cdots + u_{xx}^2]dxdt ,$$

where $\cdots$ stand for the terms u_{tt}^2 and u_{tx}^2. Recall that T_0 is characterized by (4.2) and thus the first integral in $U(T)$ tends to infinity as $T \to T_0^-$; also recall that $u \equiv v$ for $t \leq 0$. The basic strategy of the proof is the same as in [5]; we wish to show that there exist constants $\nu > 0$ ($\nu \leq \delta$ in (4.6)) and $K > 0$, independent of T, such that if

(4.8) $$|u(t,x)|^2 + |u_x(t,x)|^2 + |u_t(t,x)|^2 \leq \nu^2, \quad \nu \leq \delta ,$$

on $(-\infty,T] \times [0,1]$ then

(4.9) $$U(T) \le KF ,$$

where F is the constant defined under assumptions (f). The proof that (4.8) implies (4.9) will be outlined below using energy estimates. Once this claim is established the proof of Theorem 3.1 is completed as follows. First, by the assumptions on the history $v(t,x)$, (4.8) holds as a strict inequality for $t > 0$ sufficiently small. Next, by the Poincaré inequality (see Lemma A.1 in Appendix) and the definition of $U(T)$

(4.10) $$|u(t,x)|^2 + |u_x(t,x)|^2 + |u_t(t,x)|^2$$

$$\le \int_0^1 [u_x^2(t,x) + u_{xx}^2(t,x) + u_{tx}^2(t,x)]dx \le U(T)$$

on $(-\infty,T] \times [0,1]$. Choosing the constant $\mu^2 < \frac{\nu^2}{K}$ and $F \le \mu^2$, (4.10) shows that (4.9) implies (4.8) (as a strict inequality). Therefore if $F \le \mu^2 < \frac{\nu^2}{K}$, the estimates (4.8) and (4.9) both hold for every $T \in (-\infty,T_0)$, and consequently by Proposition 4.1 (see especially (4.2)) $T_0 = +\infty$. Moreover, (4.7) and (4.9) imply that

(4.11) $$u,u_t,u_x,u_{tt},u_{tx},u_{xx} \in L^\infty((-\infty,\infty);L^2(0,1)) \cap L^2((-\infty,\infty);L^2(0,1))) .$$

But $u,u_t,u_x,u_{tt},u_{tx},u_{xx} \in L^2((-\infty,\infty);L^2(0,1))$ also implies that

(4.12) $$u,u_t,u_x \to 0 \quad \text{in} \quad L^2(0,1) \quad \text{as} \quad t \to \infty ,$$

which in view of $u,u_t,\ldots,u_{xx} \in L^\infty((-\infty,\infty);L^2(0,1))$ yields (3.11) and completes the proof.

It remains to establish that (4.8) implies (4.9). For this purpose we will need the following properties of strongly positive kernels. Introduce the notation

$$(a*g)(t) = \int_{-\infty}^t a(t - \tau)g(\tau)d\tau$$

and

$$Q[a,w;s] = \int_{-\infty}^s w(t)(a*w)(t)dt .$$

Let assumptions (a) of Theorem 3.1 be satisfied. There exist constants $\beta, \gamma > 0$ such that for every $s \in \mathbb{R}$ and for every $w \in L^2(-\infty,s)$

$$\int_{-\infty}^{s} [(a*w)(t)]^2 dt \leq \beta Q[a,w;s], \tag{4.13}$$

where

$$\beta = \frac{1}{\alpha} \|a\|^2_{L^1(0,\infty)} + \frac{4}{\alpha} \|a'\|^2_{L^1(0,\infty)} ,$$

and

$$\int_{-\infty}^{s} [(a'*w)(t)]^2 dt \leq \gamma Q[a,w;s] , \tag{4.14}$$

where

$$\gamma = \frac{1}{\alpha} \|a'\|^2_{L^1(0,\infty)} + \frac{4}{\alpha} \|a''\|^2_{L^1(0,\infty)} .$$

The estimtes (4.13), (4.14) which have also been used in [5] are essentially contained in Staffans [20]. Another important property deals with the resolvent kernel k of the linear Volterra operator

$$\varphi'(0)y + \psi'(0)a'*y$$

defined to be the solution of the linear Volterra equation

$$\varphi'(0)k(t) + \psi'(0) \int_0^t a'(t-\tau)k(\tau)d\tau = -\psi'(0)a'(t) \qquad 0 \leq t < \infty . \tag{k}$$

<u>Lemma 4.2</u>. <u>If assumptions</u> (a) <u>are satisfied, and if</u> $\varphi'(0) > 0$, $\psi'(0) > 0$, <u>then</u> $k \in L^1(0,\infty)$.

The proof of Lemma 4.2 is given in [5].

The first estimate needed for the proof that (4.8) implies (4.9) is obtained from equation (3.5). [Recall that (3.5), (3.2), (3.3) is equivalent to the original problem (3.1)-(3.3), and it is assumption (3.4) concerning χ which will play an important role.] Multiply (3.5) by $\psi(u)_{xt}$ and integrate over $(-\infty,s] \times [0,1]$, $s < T_0$. After several integrations by parts in which the boundary condition is invoked we obtain

(4.15) $$\frac{1}{2}\int_0^1 \chi'(u(s,x))\psi'(u(s,x))u_x^2(s,x)\,dx + Q[a,s;\psi(u)_{xt}]$$

$$= -\frac{1}{2}\int_{-\infty}^{s}\int_0^1 [\chi'(u)\psi''(u) + \chi''(u)\psi'(u)]u_t u_x^2\,dx\,dt$$

$$- \frac{1}{2}\int_{-\infty}^{s}\int_0^1 \psi''(u)u_x u_t^2\,dx\,dt + \int_0^1 f(s,x)\psi(u(s,x))_x\,dx$$

$$- \int_{-\infty}^{s}\int_0^1 f_t\psi(u)_x\,dx\,dt .$$

In contrast to the analogous calculation in [5, see (3.21)], no useful information is extracted here from integration of the term $u_t\psi(u)_{xt}$ over $(-\infty,s] \times [0,1]$.

Remark 4.3. When obtaining the analogous estimates for the boundary-initial value problem (3.12) (see Remark 3.3), the analogue of equation (3.5) from which the global estimates are calculated is

$$u_t + \chi(u)_x + \int_0^t a(t-\tau)\psi(u(\tau,x))_{x\tau}\,d\tau = h(t,x) - a(t)\psi(u_0)_x .$$

To simplify several of the estimates which follow we make the additional assumption that φ,ψ (and hence also χ) $\in C^3(\mathbb{R})$; the alternative is to employ difference quotients and pass to limits as in [5]. Differentiate (3.5) with respect to t (use $(a*g)(t) = \int_0^\infty a(\xi)g(t-\xi)\,d\xi$, differentiate, and then change variables) obtaining

(4.16) $$u_{tt} + \chi(u)_{xt} + a*\psi(u)_{xtt} = f_t \qquad (-\infty < t < \infty,\ 0 \le x \le 1) .$$

Multiply (4.16) by $\psi(u)_{xtt}$ and integrate over $(-\infty,s] \times [0,1]$. After several integrations by parts the result of this tedious calculation is

$$(4.17)\qquad \frac{1}{2}\int_0^1 \chi'(u(s,x))\psi'(u(s,x))u_{xt}^2(s,x)\,dx + Q[s,s;\psi(u)_{xtt}]$$

$$= -(I_1 + \frac{1}{2} I_2 + I_3) - (J_1 + \cdots + J_7) + \int_0^1 f_t(s,x)\psi(u(s,x))_{xs}dx$$

$$- \int_{-\infty}^{s}\int_0^1 f_{tt}\psi(u)_{xt}dxdt + \frac{1}{2}\int_{-\infty}^{s}\int_0^1 [\chi''(u)\psi'(u) + \chi'(u)\psi''(u)]u_{xt}^2 u_t dxdt \ ;$$

the terms I_k in (4.17) come from

$$I = \int_{-\infty}^{s}\int_0^1 u_{tt}\psi(u)_{xtt}dxdt = I_1 + I_2 + I_3 + I_4 \ ,$$

where

$$I_1 = \int_{-\infty}^{s}\int_0^1 \psi'''(u)u_{tt}u_t^2 u_x dxdt \ ,$$

$$I_2 = \int_{-\infty}^{s}\int_0^1 \psi''(u)u_{tt}^2 u_x dxdt \ ,$$

$$I_3 = 2\int_{-\infty}^{s}\int_0^1 \psi''(u)u_{tt}u_{xt}u_t dxdt \ ,$$

$$I_4 = \int_{-\infty}^{s}\int_0^1 \psi'(u)u_{tt}u_{xtt}dxdt = -\frac{1}{2} I_2 \ ;$$

the terms J_k in (4.17) come from

$$J = \int_{-\infty}^{s}\int_0^1 \psi(u)_{xtt}\chi(u)_{xt}dxdt = J_1 + \cdots + J_7 + J_8 \ ,$$

where

$$J_8 = \int_{-\infty}^{s}\int_0^1 \chi'(u)\psi'(u)u_{xt}u_{xtt}dxdt$$

$$= \frac{1}{2}\int_0^1 \chi'(u(s,x))\psi'(u(s,x))u_{xt}^2(s,x)dx$$

$$- \frac{1}{2}\int_{-\infty}^{s}\int_0^1 [\chi''(u)\psi'(u) + \chi'(u)\psi''(u)]u_{xt}^2u_tdxdt$$

which respectively give the first term on the left side and last term on the right side of (4.17),

$$J_1 = \int_{-\infty}^{s}\int_0^1 \chi''(u)\psi'''(u)u_x^2u_t^3dxdt \ ,$$

$$J_2 = 2\int_{-\infty}^{s}\int_0^1 \chi''(u)\psi''(u)u_{xt}u_t^2u_xdxdt \ ,$$

$$J_3 = \int_{-\infty}^{s}\int_0^1 \chi''(u)\psi''(u)u_{tt}u_x^2u_tdxdt \ ,$$

$$J_4 = \int_0^1 \chi''(u(s,x))\psi'(u(s,x))u_t(s,x)u_x(s,x)u_{xt}(s,x)dx$$

$$- \int_{-\infty}^{s}\int_0^1 \chi''(u)\psi'(u)u_{xt}^2u_tdxdt - \int_{-\infty}^{s}\int_0^1 u_{xt}u_x\left[\chi''(u)\psi'(u)u_t\right]_t dxdt \ ,$$

$$J_5 = \int_{-\infty}^{s}\int_0^1 \chi'(u)\psi'''(u)u_{xt}u_xu_t^2dxdt \ ,$$

$$J_6 = 2\int_{-\infty}^{s}\int_0^1 \chi'(u)\psi''(u)u_{xt}^2u_tdxdt \ ,$$

$$J_7 = \int_{-\infty}^{s}\int_0^1 \chi'(u)\psi''(u)u_{xt}u_{tt}u_xdxdt \ .$$

The next estimate follows from the identity

$$a(0)\psi(u)_{xt} = a'*\psi(u)_{xt} + a*\psi(u)_{xtt} \tag{4.18}$$

which is derived by integrating $a*\psi(u)_{xtt}$ by parts. Multiply (4.18) by u_{xt} and

integrate over $(-\infty,s] \times [0,1]$; then in each term on the right hand side use the Cauchy Schwartz inequality and (4.13), (4.14). This leads to the estimate

$$(4.19) \qquad a(0) \int_{-\infty}^{s} \int_{0}^{1} u_{tx}\psi(u)_{xt}\,dxdt$$

$$\leq \Big[\int_{-\infty}^{s} \int_{0}^{1} u_{tx}^2\,dxdt\Big]^{1/2} \Big\{(\gamma Q[a,s;\psi(u)_{xt}])^{1/2} + (\beta Q[a,s;\psi(u)_{xtt}])^{1/2}\Big\} .$$

Using $\psi(u)_{xt} = \psi'(u)u_{xt} + \psi''(u)u_x u_t$ and (4.6) on the left side of (4.19) and $ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon}{2} b^2$, $\varepsilon = 2/\kappa a(0)$ on the right side of (4.19) gives the useful estimate

$$(4.20) \qquad \frac{\kappa a(0)}{2} \int_{-\infty}^{s} \int_{0}^{1} u_{tx}^2\,dxdt - \frac{\gamma}{\kappa a(0)} Q[a,s;\psi(u)_{xt}]$$

$$- \frac{\beta}{\kappa a(0)} Q[a,s;\psi(u)_{xtt}] \leq a(0) \int_{-\infty}^{s} \int_{0}^{1} \psi''(u)u_{xt}u_x u_t\,dxdt .$$

Next write (4.16) in the form

$$u_{tt} = -(\chi''(u)u_x u_t + \chi'(u)u_{xt} + a*\psi(u)_{xtt}) + f_t ;$$

square both sides, integrate over $(-\infty,s] \times [0,1]$, use $\left(\sum_{i=1}^{n} a_i\right)^2 \leq n \sum_{i=1}^{n} a_i^2$ $(n = 4)$, and use (4.13) to obtain the estimate

$$(4.21) \qquad \int_{-\infty}^{s} \int_{0}^{1} u_{tt}^2\,dxdt - 4 \int_{-\infty}^{s} \int_{0}^{1} [\chi'(u)]^2 u_{xt}^2\,dxdt$$

$$- 4\beta Q[a,s;\psi(u)_{xtt}] \leq 4 \int_{-\infty}^{s} \int_{0}^{1} [\chi''(u)]^2 u_x^2 u_t^2\,dxdt + 4 \int_{-\infty}^{s} \int_{0}^{1} f_t^2\,dxdt .$$

We now return to (3.1), differentiate with respect to t and obtain

$$u_{tt} = -(\varphi'(u)u_{xt} + \varphi''(u)u_x u_t + a'*\psi'(u)u_{xt} + a'*\psi''(u)u_x u_t) + f_t ,$$

square both sides, integrate with respect to x over $[0,1]$, and evaluate at $t = s$. This gives the estimate

$$(4.22)\qquad \int_0^1 u_{tt}^2(s,x)\,dx - 5\int_0^1 [\varphi'(u(s,x))]^2 u_{tx}^2(s,x)\,dx$$

$$- 5\|a'\|^2_{L^1(0,\infty)} \sup_{(-\infty,s]} \int_0^1 [\psi'(u(t,x))]^2 u_{tx}^2(t,x)\,dx$$

$$\leq 5\int_0^1 [\varphi''(u(s,x))]^2 u_x^2(s,x) u_t^2(s,x)\,dx + 5\int_0^1 f_t^2(s,x)\,dx$$

$$+ 5\|a'\|^2_{L^1(0,\infty)} \sup_{-\infty<t\leq s} \int_0^1 [\psi''(u(t,x)]^2 u_x^2(t,x) u_t^2(t,x)\,dx \,.$$

Next, differentiate (3.1) with respect to x obtaining

$$u_{tx} + \varphi(u)_{xx} + a'*\psi(u)_{xx} = f_x \,,$$

and write $\varphi(u)_{xx} = \varphi'(u)u_{xx} + \varphi''(u)u_x^2 = \varphi'(0)u_{xx} + [\varphi'(u) - \varphi'(0)]u_{xx} + \varphi''(u)u_x^2$, and similarly for $\psi(u)_{xx}$. This gives the equation

$$(4.23)\qquad \varphi'(0)u_{xx} + \psi'(0)a'*u_{xx} = X(t,x) \,,$$

where

$$X(t,x) = -u_{tx} + f_x - [\varphi'(u) - \varphi'(0)]u_{xx} - \varphi''(u)u_x^2$$
$$- a'*[\psi'(u) - \psi'(0)]u_{xx} - a'*\psi''(u)u_x^2 \,.$$

Letting k be the resolvent of the operator on the left side of (4.23) we have

$$(4.24)\qquad \varphi'(0)u_{xx}(t,x) = X(t,x) + (k*X)(t,x) \,.$$

By Lemma 4.2 $k \in L^1(0,\infty)$, and this gives the estimates

$$(4.25)\qquad [\varphi'(0)]^2 \int_0^1 u_{xx}^2(s,x)\,dx \leq 2\int_0^1 X^2(s,x)\,dx + 2\|k\|^2_{L^1} \sup_{(-\infty,s]} \int_0^1 X^2(t,x)\,dx \,,$$

and

$$(4.26)\qquad [\varphi'(0)]^2 \int_{-\infty}^s \int_0^1 u_{xx}^2\,dx\,dt \leq 2[1 + \|k\|_{L^1}] \int_{-\infty}^s \int_0^1 X^2\,dx\,dt \,.$$

To complete the proof that (4.8) implies (4.9), we first use (4.8), (4.6), (f), and the fact that for $|u(t,x)| \leq \nu$, the derivatives $|\varphi'(u(t,x))|$, $|\varphi''(u(t,x))|$,

$|\varphi'''(u(t,x))|$, $|\psi'(u(t,x))|$, $|\psi''(u(t,x))|$, $|\psi'''(u(t,x))|$, $|\chi'(u(t,x))|$, $|\chi''(u(t,x))|$, $|\chi'''(u(t,x))|$ are bounded by a constant $C > 0$ in order to simplify the basic estimates derived above; here we have used the additional simplifying assumption that φ, ψ and hence also $\chi \in C^3(\mathbb{R})$; this can be avoided as in [5].

Thus (4.15) becomes

$$\frac{\kappa^2}{4}\int_0^1 u_x^2(s,x)\,dx + Q[a,\psi(u)_{xt};s] \le \frac{3}{2}\nu C^2 \int_{-\infty}^{s}\int_0^1 u_x^2\,dx\,dt + \frac{C\nu}{2}\int_{-\infty}^{s}\int_0^1 u_t^2\,dx\,dt + FC\Big[\frac{C}{\kappa^2} + \frac{1}{2C^2\sqrt{\nu}}\Big], \tag{4.27}$$

similarly (4.17) simplifies to

$$\begin{aligned}\frac{\kappa^2}{4}\int_0^1 u_{xt}^2(s,x)\,dx &+ Q[a,\psi(u)_{xtt};s] \\ &\le C^2\nu\int_0^1 u_{xt}^2(s,x)\,dx + C\nu\Big[\frac{1}{2} + C\Big]\int_0^1 u_x^2(s,x)\,dx \\ &+ C\nu[F + C + C\nu + C\nu^2]\int_{-\infty}^{s}\int_0^1 u_x^2\,dx\,dt + C^2\nu^2\int_{-\infty}^{s}\int_0^1 u_t^2\,dx\,dt \\ &+ C\nu\Big[\frac{3}{2} + \Big(\frac{9}{2} + \frac{7}{2}\nu\Big)C\Big]\int_{-\infty}^{s}\int_0^1 u_{tx}^2\,dx\,dt \\ &+ C\nu\Big[2 + \frac{5}{2}C + \Big(\frac{1}{2} + C\Big)\nu\Big]\int_{-\infty}^{s}\int_0^1 u_{tt}^2\,dx\,dt + CF\Big(\nu + \frac{1}{\kappa^2} + \frac{1}{2\nu}\Big);\end{aligned} \tag{4.28}$$

the estimate (4.20) becomes

$$(4.29)\qquad \frac{\kappa a(0)}{2}\int_{-\infty}^{s}\int_{0}^{1} u_{tx}^{2}\,dxdt - \frac{\gamma}{\kappa a(0)}\,Q[a,s;\psi(u)_{xt}]$$

$$- \frac{\beta}{\kappa a(0)}\,Q[a,s;\psi(u)_{xtt}] \le a(0)\,\frac{\nu C}{2}\left(\int_{-\infty}^{s}\int_{0}^{1} u_{xt}^{2}\,dxdt + \int_{-\infty}^{s}\int_{0}^{1} u_{x}^{2}\,dxdt\right);$$

estimate (4.21) simplifies to

$$(4.30)\qquad \int_{-\infty}^{s}\int_{0}^{1} u_{tt}^{2}\,dxdt - 4C^{2}\int_{-\infty}^{s}\int_{0}^{1} u_{xt}^{2}\,dxdt$$

$$- 4\beta Q[a,s;\psi(u)_{xtt}] \le 4C^{2}\nu^{2}\int_{-\infty}^{s}\int_{0}^{1} u_{t}^{2}\,dxdt + 4F;$$

estimate (4.22) simplifies to

$$(4.31)\qquad \int_{0}^{1} u_{tt}^{2}(s,x)\,dx - 5C^{2}\int_{0}^{1} u_{tx}^{2}(s,x)\,dx$$

$$- 5C^{2}\|a'\|_{L^{1}} \sup_{-\infty<t\le s}\int_{0}^{1} u_{tx}^{2}(t,x)\,dx \le 5C^{2}\nu^{2}\int_{0}^{1} u_{x}^{2}(s,x)\,dx$$

$$+ 5C^{2}\nu^{2}\|a'\|_{L^{1}} \sup_{-\infty<t\le s}\int_{0}^{1} u_{x}^{2}(t,x)\,dx + 5F;$$

the estimate (4.25) becomes

$$(4.32)\qquad \kappa^{2}\int_{0}^{1} u_{xx}^{2}(s,x)\,dx \le 2\int_{0}^{1} X^{2}(s,x)\,dx + 2\|k\|_{L^{1}}^{2} \sup_{(-\infty,s]}\int_{0}^{1} X^{2}(t,x)\,dx;$$

finally, (4.26) becomes

$$(4.33)\qquad \kappa^{2}\int_{-\infty}^{s}\int_{0}^{1} u_{xx}^{2}\,dxdt \le 2[1 + \|k\|_{L^{1}}]\int_{-\infty}^{2}\int_{0}^{1} X^{2}\,dxdt,$$

where in the estimates (4.32) and (4.33) $X(t,x)$ is the function (depending on u, u_x, u_{xx} u_{tx}, a, and f) given preceding formula (4.24).

We now focus our attention on the simplified estimates (4.27) through (4.33).

By the Poincaré inequality (see Appendix) $U(T)$, given by (4.7), can be majorized by

$$(4.34) \qquad U(T) \le \sup_{(-\infty,T]} \Big[\int_0^1 [2u_x^2(t,x) + u_{tt}^2(t,x) + 2u_{tx}^2(t,x) + u_{xx}(t,x)]\Big]dx$$

$$+ \int_{-\infty}^{T} \int_0^1 [u_{tt}^2 + 2u_{tx}^2 + 3u_{xx}^2]\,dxdt \ .$$

Moreover, each term on the right hand side of (4.34) can be majorized by a suitable linear combination of the left hand sides of the estimates (4.27-4.33). On the other hand, each term on the right hand sides of the estimates (4.27-4.33) can be majorized by terms of the form $O(F)$, or $O(\nu)U(T)$, or $\varepsilon U(T) + c(\varepsilon)F(T)$ for any $\varepsilon > 0$; the last of these comes from estimating the right hand sides of (4.32) and (4.33). This combined with (4.34) yields the final estimate

$$(4.35) \qquad U(T) \le \{O(\nu) + O(\varepsilon)\}U(T) + c(\varepsilon)F \ .$$

Therefore, fixing $\nu > 0$, $\varepsilon > 0$ sufficiently small, (4.35) yields (4.9), assuming that (4.8) holds. This completes the proof of Theorem 3.1.

5. Development of Singularities. We consider the pure Cauchy problem

$$(5.1)\qquad \begin{cases} u_t + \varphi(u)_x + a'*\psi(u)_x = 0 \\ u(0,x) = u_0(x) \qquad (x \in \mathbb{R}) \end{cases};$$

throughout this section $*$ will denote the convolution on $[0,t]$ (not $(-\infty,t]$). For a discussion of the existence of smooth solutions of (5.1) on $[0,\infty) \times \mathbb{R}$ for sufficiently smooth and "small" data we refer the reader to Remarks 3.3, 3.6. It is known (see Proposition 4.1 and Remark 4.2) that if a, φ, ψ satisfy the assumptions of Proposition 4.1 and $u_0 \in H^2(\mathbb{R})$, then there exists a unique smooth solution of (5.1) on a maximal interval $[0,T_0) \times \mathbb{R}$, $0 < T_0 \leq \infty$.

Our objective is to study the problem of the development of singularities (shocks) of the solution in finite time such that a physically meaningful entropy condition will be satisfied (see [14]), assuming that a local smooth solution exists.

Let $\xi \in \mathbb{R}$ and let $u(t,x)$ be a smooth solution of (5.1). We define the characteristic through ξ of (5.1) to be the curve $x = x(t,\xi)$ in the t,x plane specified by the initial value problem

$$(5.2)\qquad \begin{cases} \dfrac{dx}{dt} = \varphi'(u(t,x)) \qquad (t > 0) \\ x(0,\xi) = \xi \qquad (\xi \in \mathbb{R}) . \end{cases}$$

It should be noted that the total derivative of $u(t,x)$ along the characteristic $x(t,\xi)$ is

$$\begin{aligned} \frac{d}{dt} u(t,x(t,\xi)) &= u_t(t,x(t,\xi)) + u_x(t,x(t,\xi)) \frac{dx}{dt} \\ &= u_t(t,x(t,\xi)) + \varphi'(u(t,x(t,\xi)))u_x(t,x(t,\xi)) \\ &\overset{\text{def}}{=} u_t(t,x(t,\xi)) + \varphi(u(t,x(t,\xi)))_x . \end{aligned}$$

Let $u(t,\xi) = u(t,x(t,\xi))$. If u is a solution of (5.1) its derivative along the characteristic $x(t,\xi)$ satisfies the integrodifferential equation

$$
(5.3) \qquad \begin{cases} \dfrac{du}{dt} = -\displaystyle\int_0^t a'(t-\tau)\psi(u(\tau,x(t,\xi)))_x d\tau & (t > 0) \\ \\ u(0,x(0,\xi)) = u_0(x(0,\xi)) = u_0(\xi) & (\xi \in R) . \end{cases}
$$

The reader should note that (5.3) is no longer of convolution type, because of the term $\psi(u(\tau,x(t,\xi)))_x = \psi'(u(\tau,x(t,\xi)))u_x(\tau,x(t,\xi))$ under the integral.

Let $x(t,\xi)$ be the characteristic of (5.1) through ξ and define $v(t,\xi) = \frac{\partial}{\partial \xi} x(t,\xi)$; note that $v(0,\xi) = 1$. The function v measures the growth of the characteristics with respect to ξ. Let $\varphi''(\cdot) \neq 0$. According to Lemma 2.1 of [14] a singularity will develop in the solution u of (5.1) in finite time, if it can be shown that there exists a number $\bar{T}$, $0 < \bar{T} < \infty$ such that $v(\bar{T},\xi) < 0$. For, in this case there exists a $0 < T < \bar{T}$ and $\xi_1 \neq \xi_2 \in R$ such that $x(T,\xi_1) = x(T,\xi_2)$ (i.e. the characteristics through ξ_1 and ξ_2 cross at T), and $u(\bar{T}^-,x(\bar{T},\xi_1)) \neq u(\bar{T}^-,x(\bar{T},\xi_2))$. This is the definition of the development of a "shock" at $\bar{T}$ in the solution u of (5.1). It is explained in [14] that this "shock" solution satisfies the physically meaningful entropy condition.

We shall therefore set up a differential equation for $v(t,\xi)$. Let $w(t,\xi) = \frac{\partial u}{\partial \xi}(t,x(t,\xi))$, and note that $w(0,\xi) = u_0'(\xi)$. Then using (5.2)

$$
\frac{dv}{dt} = \frac{d}{dt} x_\xi(t,\xi) = \frac{\partial}{\partial \xi}\frac{dx}{dt} = \frac{\partial}{\partial \xi}\varphi'(u(t,x(t,\xi)))
$$

$$
= \varphi''(u(t,x(t,\xi)))w(t,\xi) .
$$

Thus v satisfies the initial value problem

$$
(5.4) \qquad \frac{dv}{dt} = \varphi''(u(t,x(t,\xi)))w, \quad v(0,\xi) = 1 .
$$

The equation satisfied by w is found by differentiating (5.3) with respect to ξ, obtaining the initial value problem

$$
(5.5) \qquad \frac{dw}{dt} = -v(t,\xi)\int_0^t a'(t-\tau)[\psi''(u(\tau,x(t,\xi)))u_x^2(\tau,x(t,\xi))
$$

$$
+ \psi'(u(\tau,x(t,\xi)))u_{xx}(\tau,x(t,\xi))]d\tau, \quad w(0) = u_0'(\xi) .
$$

Our objective is to use the system of four nonlinear equations (5.2)-(5.5) for the quantities $x(t,\xi)$, $u(t,\xi)$, $v(t,\xi)$, $w(t,\xi)$ satisfying the indicated initial conditions to establish the development of a shock in the sense described above. This problem which is under active study remains to be solved in this generality.

We restrict ourselves to the special case $\psi \equiv \varphi$ in (5.1). By Remarks 3.5, 3.6 the Cauchy problem (5.1) has a unique smooth solution u on $[0,\infty) \times \mathbb{R}$ if a and φ satisfy the assumptions of Theorem 3.1, $0 < a(0) < 1$, and if $\|u_0\|_{H^2(\mathbb{R})}$ is sufficiently small. In the special case $\psi = \varphi$ we can use the method of MacCamy [11] and Dafermos and Nohel [4], introduce the resolvent kernel k of a' defined by the equation

$$\text{(k)} \qquad k(t) + (a'*k)(t) = -a'(t) \quad (0 \le t < \infty) ,$$

and write (5.1) in the equivalent form

$$\text{(5.6)} \qquad \begin{cases} u_t + \varphi(u)_x + k(0)u + k'*u = k(t)u_0(x) & (0 < t < \infty,\ x \in \mathbb{R}) \\ u(0,x) = u_0(x) & (x \in \mathbb{R}) . \end{cases}$$

Note that since a satisfies assumptions (a), $k(0) = -a'(0) > 0$. The method of Lemma 4.2 applied to equation (k) shows that since $0 < a(0) < 1$, one has

$$\text{(5.7)} \qquad k, k' \in L^1(0,\infty) .$$

Remark. (5.7) also holds if $a(t) = a_\infty + A(t)$, $a(0) = 1$, $a_\infty > 0$, and A satisfies assumptions (a).

To establish the development of singularities in a smooth solution u of (5.1) with $\psi \equiv \varphi$ (equivalently of (5.6)) we study the system of nonlinear equations corresponding to (5.2)-(5.5). In this case it is easily seen that the quantities $x(t,\xi)$, $u(t,\xi) = u(t,x(t,\xi))$, $v(t,\xi) = \frac{\partial}{\partial\xi} x(t,\xi)$, $w(t,\xi) = \frac{\partial}{\partial\xi} u(t,x(t,\xi))$ satisfy the initial value problem

$$
(5.8)\qquad \begin{cases}
\dfrac{dx}{dt} = \varphi'(u(t,x(t,\xi))) \\[2ex]
\dfrac{du}{dt} + k(0)u + \displaystyle\int_0^t k'(t-\tau)u(\tau,x(t,\xi))d\tau = k(t)u_0(x) \\[2ex]
\dfrac{dv}{dt} = \varphi''(u(t,x(t,\xi))) \\[2ex]
\dfrac{dw}{dt} + k(0)w + \left(\displaystyle\int_0^t k'(t-\tau)u_x(\tau,x(t,\xi))d\tau - u_0'(t,x(t,\xi))\right)v = 0 \\[2ex]
x(0,\xi) = \xi,\ u(0,\xi) = u_0(\xi),\ v(0,\xi) = 1,\ w(0,\xi) = u_0'(\xi) .
\end{cases}
$$

In recent joint work with Malek-Madani [15] we have established the following result; its proof which uses (5.7), (5.8) and the general strategy for the formation of shocks outlined above will appear elsewhere. A similar result was stated by MacCamy in his lecture; see note in [6].

Theorem 5.1. Let a $(0 < a(0) < 1)$ satisfy asumptions (a). In addition, let the resolvent kernel k of a' satisfy

(5.9) $\qquad k(t) \geq 0, \quad k'(t) \leq 0 \qquad (0 \leq t < \infty)$.

Let $\varphi \in C^2(\mathbb{R})$, $\varphi(0) = 0$, $\varphi'(\cdot) > 0$, $\varphi''(\cdot) > 0$. Let $u_0 \in C^2(\mathbb{R})$. If $u_0'(\xi) < 0$ and $|u_0'(\xi)|$, $\xi \in \mathbb{R}$, is sufficiently large, every (necessarily) smooth solution $u(t,x)$ of the Cauchy problem (5.6) (<===> (5.1) if $\psi \equiv \varphi$) will develop a shock in finite time. If $\varphi''(\cdot) \geq \beta > 0$, an upper bound for the time at which a shock develops is

$$
\bar{T} = \frac{1}{k(0)} \log \frac{u_0'(\xi)\beta}{u_0'(\xi)\beta + k(0)} .
$$

The following considerations provide examples of kernels a in (5.1) (<===> (5.6) if $\psi \equiv \varphi$) for which Theorem 5.1 can be applied.

Remark 5.2. If $a(t) = \beta e^{-\alpha t}$ $(0 < \beta < 1)$, a simple calculation shows that $k(t) = \alpha\beta e^{-\alpha(1-\beta)t}$, and evidently the inequalities (5.9) are satisfied in the strict sense.

More generally, one has the following result established by elementary consideration from the resolvent equation (k).

Lemma 5.3. Let $a \in C^2[0,\infty)$, $(-1)^j a^{(j)}(t) \geq 0$, $j = 0,1,2$ $(0 \leq t < \infty)$, and assume that

$$a'(t)a'(0) - a''(t) < 0 \qquad (0 \leq t < \infty) .$$

Then $k(t) > 0$, $k'(t) < 0$ $(0 \leq t < \infty)$. If also $0 < a(0) < 1$, and $a \in W^{2,1}[0,\infty)$, then $k, k' \in L^1(0,\infty)$.

Corollary 5.4. Let $a(t) = \sum_{j=1}^{m} \beta_j e^{-\alpha_j t}$, $\beta_j > 0$, $\alpha_j > 0$; if

$$\alpha_i > \sum_{j=1}^{m} \alpha_j \beta_j \qquad (i = 1,2,\ldots,m) ,$$

then $k(t) > 0$, $k'(t) < 0$ $(0 \leq t < \infty)$; if also $\sum_{j=1}^{m} \beta_j < 1$ then $k, k' \in L^1(0,\infty)$.

Finally we remark that the general approach used to prove Theorem 5.1 can also be used to show that if by contrast, $u_0'(\xi) > 0$, and if the other assumptions of Theorem 5.1 hold, no singularities develop in the solution u in finite time. Thus Theorem 5.1, together with this remark, form the analogue for the conservation law with memory (5.1) of Lax's classical result for the conservation law (1.2).

Appendix

For the convenience of the reader we state and prove the following elementary inequalities which were used in the proof of Theorem 3.1 and which are generally referred to as Poincaré inequalities.

Lemma A.1. Let $g, g' \in L^2(0,1)$, g real, and let $\bar{g} = \int_0^1 g(x)dx$. Then

$$\int_0^1 g^2(x)dx \leq \bar{g}^2 + \int_0^1 [g'(x)]^2 dx \; ;$$

in particular, if $\bar{g} = 0$, then

$$\int_0^1 g^2(x)dx \leq \int_0^1 [g'(x)]^2 dx \; .$$

Proof. Take $0 \leq x_0 < x \leq 1$. Then

$$g(x) - g(x_0) = \int_{x_0}^{x} g'(\xi)d\xi \; ,$$

and by Cauchy-Schwartz

$$(g(x) - g(x_0))^2 \leq (x - x_0) \int_{x_0}^{x} (g'(\xi))^2 d\xi \leq \int_0^1 (g'(\xi))^2 d\xi \; .$$

Thus

$$\int_0^1 (g(x) - g(x_0))^2 dx = \int_0^1 g^2(x)dx - 2g(x_0)\bar{g} + g^2(x_0)$$

$$\leq \int_0^1 [g'(x)]^2 dx \; .$$

By the continuity of g choose x_0 such that $g(x_0) = \bar{g}$, and the first inequality is immediate.

<u>Lemma A.2.</u> <u>Let</u> $g, g', g'' \in L^2(0,1)$, g real, $g(1) = g(0)$, $\bar{g} = \int_0^1 g(x)dx = 0$. <u>Then</u>

$$g^2(x) + [g'(x)]^2 \le \int_0^1 [g''(x)]^2 dx \qquad (0 \le x \le 1) .$$

<u>Proof.</u> Let $0 \le y < x \le 1$; we have

$$g'(x) - g'(y) = \int_y^x g''(\xi)d\xi .$$

Squaring both sides and using Cauchy-Schwartz gives

$$[g'(x)]^2 + [g'(y)]^2 - 2g'(x)g'(y) \le \int_0^1 [g''(x)]^2 dx .$$

Integrating with respect to y over $[0,1]$ and using $\int_0^1 g'(y)dy = g(1) - g(0) = 0$ we have

$$[g'(x)]^2 + \int_0^1 [g'(y)]^2 dy \le \int_0^1 [g''(x)]^2 dx .$$

Since $\bar{g} = 0$ the conclusion follows from the inequality

$$[g(x)]^2 \le \int_0^1 [g'(y)]^2 dy ,$$

the proof of which is contained in that of Lemma 1.

<u>Application.</u> If $w \in X(M,T)$ defined in the proof of Proposition 4.1, then

$$w^2(t,x) + w_x^2(t,x) + w_t^2(t,x) \le \int_0^1 [w_{xx}^2(t,x) + w_{tx}^2(t,x)]dx \quad (0 \le x \le 1) .$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we study a history-boundary value problem for a nonlinear conservation with fading memory in one space dimension. The motivation for studying this problem is an earlier work by C. M. Dafermos and the author concerning the motion of a nonlinear, one-dimensional viscoelastic body. Using a variant of an energy method applied to the viscoelastic problem it is shown

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20. ABSTRACT - Cont'd.

that under physically reasonable assumptions the nonlinear conservation law has a unique, classical solution (global in time), provided the data are sufficiently smooth and "small" in a suitable norm; moreover, the solution and its first order derivatives decay to zero as $t \to \infty$. The proof illustrates the versatility of the energy method combined with frequency domain techniques for Volterra operators.

A preliminary analysis based on current work of R. Malek-Madani and the author is presented concerning the development of singularities in smooth solutions of the conservation law (in finite time) for sufficiently "large" smooth data; under special assumptions it is shown that such singularities necessarily develop. The hope is to apply such a procedure to the visco-elastic problem.

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